Calculus: Differentiability and derivatives

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5 Differentiability and derivatives

In this chapter we study another fundamental concept for infinitesimal calculus and mathematics in general: the differentiability of functions and its prime application: the derivative.

Note: The formal separation into Chapters 5 and 6 (as by the Course Outline) is lifted and we present differentiability and derivatives together.

5.1 Differentiability in a point

This part of calculus has been developed in the 17th century when mathematicians and scientists were studying the motion of bodies, with applications from engineering (e.g., projectiles) to astronomy. In that context it is necessary to represent the position $x(t)$, velocity $v(t)$ and acceleration $a(t)$ of a body as functions of time and clarify the relation between these quantities.

The questions addressed can be illustrated with the following example. As we know,

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

is the *average velocity* of a body that has traveled some distance Δx during an interval of time Δt. Of course, we know from experience that in general a body can move with varying velocity. Hence, we propose to define the *instantaneous velocity* as the limit of the average velocity as the time interval tends to zero:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}.$$

Another application of differential calculus is to find the extremal values of functions and the curves that represent these functions. We will come back to this topic later.

First, we will introduce the concept of the derivative at a point, as illustrated in Fig. 1. We consider the function $f(x)$ and the right triangle APQ. The point A is located at $(x, y) = (a, f(a))$. The point P is located at $(x, y) = (a + \Delta x, f(a + \Delta x))$, i.e., represents another point on the curve of $f(x)$, separated from A on the abscissa by a distance Δx. The straight line that cuts the function $f(x)$ in A and P represents a *secant* S of the curve. The point Q is located at $(x, y) = (a+\Delta x, f(a))$ by construction of the right triangle.

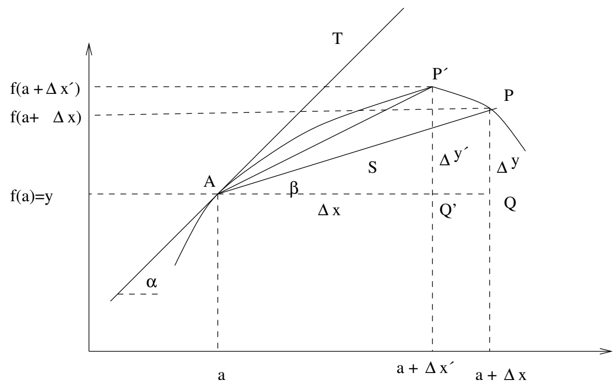

Figure 1: Construction of the tangent T to the function at a point A.

It is possible to determine the angle β of the secant with the abscissa using elementary geometry:

$$\tan \beta = \frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x},$$

where Δy is the difference of the values of $f(x)$ in A and P. The expression $\dfrac{\Delta y}{\Delta x} = \dfrac{f(a + \Delta x) - f(a)}{\Delta x}$ is also known as *difference quotient* and geometrically represents the ratio between the opposite and adjacent of the right triangle APQ. Obviously, there is nothing special about the point P and we can choose any other point P' on the curve, with other differences $\Delta x'$ and $\Delta y'$.

We want to respond to the following question: Can we move the point P' towards A and if so, (a) what value of the angle with the abscissa do we obtain and (b) what does the straight line represent in this case?

Using limits, we can reformulate:

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \tan \beta = \tan \alpha,$$

where α is the angle in A of the curve with the abscissa, and the straight line now represents the *tangent* T of the curve in A. Geometrically, the limit represents the slope of the tangent.

Definition 1 (derivative in a point): If the limit

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

exists, then f is *differentiable* in $x = a$ and the limit is the *derivative of* f in a and we write it $f'(a)$ or $\dfrac{df}{dx}(a)$ or $\dfrac{df(a)}{dx}$.

Comments:
(1) We note that by definition $f(a)$ has to exist (like for a continous function but unlike for the existence of a limit of f in a). This is intuitively clear as otherwise there would be no point $(a, f(a))$ to construct a tangent at.

(2) Similar to continuity, differentiability relies on the existence of limits. If the limit of the difference quotient does not exist, the function does not have a derivative in this point.

(3) As the existence of a limit implies the existence of the lateral limits (and therefore includes the cases $\Delta x > 0$ and $\Delta x < 0$), there is no need to consider alternative formulations in Def. 1 including terms like $f(a - \Delta x) - f(a)$. However, observe the concept of lateral differentiability (see below).

(4) An alternative formulation of the derivative is (using $\Delta x = h$)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(5) Yet another formulation of the derivative is (using $\Delta x = x - a$)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In this formulation, the lateral limits correspond to limits $x \to a\pm$. It is possible to formulate the theorems and later examples in any of the given representations.

(6) The straight line $y(x)$ which is the tangent to f in the point A can be found starting from the equation of a straight line of slope m (already identified with $f'(a)$) and t:

$$\begin{aligned} y(x) &= m \cdot x + t, \\ y(x) &= f'(a) \cdot x + t, \end{aligned}$$

For $x = a$ we have

$$f(a) = f'(a) \cdot a + t$$

and hence $t = f(a) - f'(a) \cdot a$. Inserting t into the general expression, we obtain

$$y(x) = f(a) + f'(a) \cdot (x - a)$$

as equation of the tangent in A.

(7) The derivative $f'(a)$ represents a real number, a scalar. Later, we generalize f' to be a function of x.

(8) In some contexts, the terms df and dx are called *differentials*, and are interpreted as infinitesimal limits of f and x, in analogy to their finite counterparts Δy and Δx.

Example 1: Determine the derivative of $f(x) = b$, $b \in \mathbb{R}$, in a point a using the definition of the derivative.

We calculate:

$$f'(a) \;=\; \lim_{\Delta x \to 0} \frac{b-b}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

The zero in the numerator is an exact number and hence the result is exactly zero in spite of $\Delta x \to 0$ in the denominator. This is not an indetermination.

Example 2: Determine the derivative of $f(x) = bx$, $b \in \mathbb{R} \setminus \{0\}$, in a point a using the definition of the derivative.

We calculate:

$$f'(a) = \lim_{\Delta x \to 0} \frac{b(a+\Delta x) - ba}{\Delta x} = \lim_{\Delta x \to 0} b \cdot \frac{(a+\Delta x) - a}{\Delta x} = b \cdot \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = b \cdot \lim_{\Delta x \to 0} 1 = b,$$

where we have cancelled $\Delta x / \Delta x$ because $\Delta x \neq 0$.

Example 3: Determine the derivative of $f(x) = bx^2$, $b \in \mathbb{R} \setminus \{0\}$, in a point a using the definition of the derivative.

We calculate:

$$\begin{aligned}
f'(a) &= \lim_{\Delta x \to 0} \frac{b(a+\Delta x)^2 - ba^2}{\Delta x} = b \lim_{\Delta x \to 0} \frac{(a+\Delta x)(a+\Delta x) - a^2}{\Delta x} \\
&= b \lim_{\Delta x \to 0} \frac{(a+\Delta x)a + (a+\Delta x)\Delta x - a^2}{\Delta x} \\
&= b \lim_{\Delta x \to 0} \left(\frac{(a+\Delta x)a - a^2}{\Delta x} + \frac{(a+\Delta x)\Delta x}{\Delta x} \right).
\end{aligned}$$

We know that the limit of a sum is the sum of the limits if the limits of each term exist. We cannot confirm that yet, but we keep calculating assuming it. Therefore,

$$\begin{aligned} f'(a) &= b\lim_{\Delta x\to 0}\frac{(a+\Delta x)a-a^2}{\Delta x}+b\lim_{\Delta x\to 0}\frac{(a+\Delta x)\Delta x}{\Delta x} \\ &= b\lim_{\Delta x\to 0}a\frac{a+\Delta x-a}{\Delta x}+b\lim_{\Delta x\to 0}\frac{(a+\Delta x)\Delta x}{\Delta x} \\ &= b\lim_{\Delta x\to 0}a\frac{\Delta x}{\Delta x}+b\lim_{\Delta x\to 0}\frac{(a+\Delta x)\Delta x}{\Delta x}. \end{aligned}$$

We again are able to cancel $\Delta x/\Delta x$ and we obtain:

$$f'(a) = b\lim_{\Delta x\to 0}a + b\lim_{\Delta x\to 0}(a+\Delta x) = b(a+a) = 2ab.$$

In the last step we see that the two limits exist separately and therefore their sum exists as well.

Example 4: Determine the derivative of $f(x) = 1/x$, in a point $a \neq 0$ using the definition of the derivative.

We calculate:

$$\begin{aligned} f'(a) &= \lim_{\Delta x\to 0}\frac{\left(\dfrac{1}{a+\Delta x}-\dfrac{1}{a}\right)}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{a-(a+\Delta x)}{a(a+\Delta x)}\right) \\ &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{-\Delta x}{a(a+\Delta x)}\right) = \lim_{\Delta x\to 0}\frac{-1}{a^2+a\Delta x} \\ &= \frac{-1}{a^2}. \end{aligned}$$

In this example, we have used the same arguments as before. The result also reminds us that the value $a = 0$ is excluded. In the examples we recognize clearly the general rule of derivatives of a potential function $f(x) = x^n$:

$$f'(x) = nx^{n-1}.$$

We leave its proof as an exercise.

In analogy to limits and continuous functions, we can define also lateral differentiability.

Definition 2 (lateral differentiability):
A function f is:

$$\begin{aligned} &\text{left differentiable in } x=a \qquad \text{if and only if } \lim_{\Delta x\to 0-}\frac{f(a+\Delta x)-f(a)}{\Delta x} \quad \text{exist,} \\ &\text{right differentiable in } x=a \qquad \text{if and only if } \lim_{\Delta x\to 0+}\frac{f(a+\Delta x)-f(a)}{\Delta x} \quad \text{exist.} \end{aligned}$$

An inmediate consequence of this definition is the following theorem, in analogy to corresponding theorems for limits and continuous functions.

Theorem 1 (differentiability and lateral differentiability):
A function is differentiable in a point if and only if it is left and right differentiable.

Now, we establish a central result for real functions:

Theorem 2 (differentiability implies continuity):
Let f be differentiable in $x = a$. Then, f is continuous in $x = a$.

We want to prove this theorem. Recall that continuity of f is denoted as

$$\lim_{x \to a} f(x) = f(a).$$

Our intention is to reach that equation as *conclusion*, starting from the *hypothesis* that f is differentiable in a. We will only use simple algebraic transformations and hence it is beneficial to use the representation

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

of a function differentiable in a.

Proof:
For a function differentiable in a, $f(a)$ exists and a simple transformation permitted for $x \neq a$ is

$$f(x) - f(a) = (f(x) - f(a)) \left(\frac{x - a}{x - a} \right) = \frac{f(x) - f(a)}{x - a}(x - a).$$

We take the limit to both sides of the equation (as we take the limit $x \to a$, values $x = a$ are excluded):

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a}(x - a) \right).$$

The limit of a product is the product of the limits (and both exist). As f by hypothesis is differentiable in a, the limit $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is $f'(a)$. The limit $\lim\limits_{x \to a}(x - a)$ also exists and is equal to zero. Therefore,

$$\lim_{x \to a}(f(x) - f(a)) = f'(a) \lim_{x \to a}(x - a) = f'(a) \cdot 0 = 0.$$

The limit of a sum is the sum of the limits, if both exist:

$$\lim_{x \to a} f(x) - \lim_{x \to a} f(a) = 0,$$

and as $f(a)$ is a constant (and as such its limit exists), we have:

$$\lim_{x \to a} f(x) = f(a).$$

This equation establishes that the limit of $f(x)$ when x tends to a exists and is identical to $f(a)$. This confirmation is nothing else than the definition of a function continuous in a and we have concluded the proof.

The converse of the theorem is not true, i.e., the continuity in a point does not imply differentiability. To show this, a counterexample is enough. Consider $f(x) = |x|$, a continuous function in $\mathbb{R}$ and specifically in $a = 0$. We calculate the lateral limits there:

$$\begin{aligned}
\lim_{\Delta x \to 0-} \frac{f(a + \Delta x) - f(a)}{\Delta x} &= \lim_{\Delta x \to 0-} \frac{-(0 + \Delta x) - 0}{\Delta x} = -1, \\
\lim_{\Delta x \to 0+} \frac{f(a + \Delta x) - f(a)}{\Delta x} &= \lim_{\Delta x \to 0+} \frac{(0 + \Delta x) - 0}{\Delta x} = 1.
\end{aligned}$$

As the two lateral limits are different, the function is not differentiable in $a = 0$.

5.2 Differentiable functions

Until now, we have only considered the derivative at a point $x = a$, whose result is a scalar number. However, it is straightforward to perform the limit for all $x \in$dom(f). In fact, as there was nothing particular about a, it is possible to replace simply a by x in the calculated derivatives and obtain a derivative *function*.

In Example 3, we determined the derivative of $f(x) = bx$, $b \in \mathbb{R} \setminus \{0\}$, at a point $x = a \in \mathbb{R}$ as $f'(a) = 2ab$. Now, we replace a by x and obtain

$$f'(x) = 2bx$$

for all $x \in \mathbb{R}$. The domain of f' is a subset of the domain of f (recall that the definition of subset includes the set itself: $\mathbb{R} \subseteq \mathbb{R}$). Obviously, there is the possibility that a function is not differentiable for all $x \in$ dom(f) and then we would find that the domain of f' is a *true subset* of the domain of f: dom$(f') \subset$ dom(f).

Definition 3 (derivative function):
The derivative function f' (or short, *derivative*) of a function f is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

for the values of $x \in$ dom$f$ for which the limit exists. We also denote it as $\dfrac{df}{dx}(x)$ or $\dfrac{df(x)}{dx}$.

Comments:
(1) The domain of f' can be open, closed, or half-open intervals. A typical statement is: "Let f be differentiable in (a, b)", implying that $\mathrm{dom}(f') = (a, b)$. If the interval is closed or half-open, then for example "Let f be differentiable in $[a, b]$" means that f is differentiable in (a, b) and left differentiable in a and right differentiable in b.

(2) Definitions 1 and 3 tell us that Δx is an arbitrary non-zero quantity introduced with the sole purpose to perform a limit. It is independent of a, x or f.

(3) Once the derivative has been obtained, it can be studied alongside the original function, for example, graphically.

5.3 Properties of derivatives

The fundamental functions discussed in Theorem 3 (Chapter 4) are differentiable in their domains (consider, however, the comments made below Def. 3). But now, we have to study how arithmetic operations work for derivatives.

Theorem 3 (basic rules for derivatives):
Let f and g be differentiable functions. Then,
(i) Sum rule:
$$(f(x) + g(x))' = f'(x) + g'(x);$$
(ii) Multiplication with a constant: For any $k \in \mathbb{R}$,
$$(k\, f(x))' = k\, f'(x);$$
(iii) Product rule:
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x);$$
(iv) Quotient rule: if $g(x) \neq 0$, then
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$
Comment:
It is always important to take into account the domains of the functions: For example, if $f(x) = x^2$ with domain $\mathrm{dom}(f) = (1, 5)$ and $g(x) = x^3$ with domain $\mathrm{dom}(g) = (2, \infty)$, their derivatives $f'(x)$ and $g'(x)$ inherit the respective domains and the domain of $(f(x) + g(x))'$ is their intersection: $\mathrm{dom}(f + g) = (2, 5)$. This example also shows that the domain of f' is the domain of f although the derivative function $2x$ in principle permits x values of all $\mathbb{R}$.

We will only prove the product rule:
We use the definition:
$$(f(x)g(x))' \;=\; \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x}.$$

We want to obtain a symmetric expression in the limit and therefore we subtract and add in the numerator the same term $f(x)g(x+\Delta x)$, to obtain:

$$\begin{aligned}
(f(x)g(x))' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]g(x+\Delta x) + f(x)[g(x+\Delta x) - g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{[f(x+\Delta x) - f(x)]g(x+\Delta x)}{\Delta x} + \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x} \right).
\end{aligned}$$

The limit of the sum is the sum of the limits (if both exist), and the limit of a product is the product of the limits (if both exist). Hence,

$$(f(x)g(x))' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \lim_{\Delta x \to 0} g(x+\Delta x) + \lim_{\Delta x \to 0} f(x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}.$$

As g is differentiable, it is also continuous (Theorem 2) and therefore it is possible to interpret $g(x+\Delta x)$ as a composite function and as such $\lim_{\Delta x \to 0} g(x+\Delta x) = g(\lim_{\Delta x \to 0}(x+\Delta x)) = g(x)$. Furthermore, $\lim_{\Delta x \to 0} f(x) = f(x)$. In the next step we get:

$$\begin{aligned}
(f(x)g(x))' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x) + f(x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x)g(x) + f(x)g'(x),
\end{aligned}$$

where we have recognized the definitions of the derivatives of $f(x)$ and $g(x)$ to conclude the proof.

Example 5: Calculate the derivatives of the following functions:

(a) $f(x) = x^2\cos(x) + x^3$:
$f'(x) = 2x\cos(x) - x^2\sin(x) + 3x^2 = 2x\cos(x) + x^2(3 - \sin(x))$.

(b) $f(x) = \dfrac{e^x}{5x+1}$:
$f'(x) = \dfrac{e^x(5x+1) - 5e^x}{(5x+1)^2} = \dfrac{e^x(5x-4)}{(5x+1)^2}$.

Theorem 4 (derivative of a composite function):
Let f and g be differentiable functions. Then, the composite function $f(g(x))$ is differentiable and its derivative is

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

Theorem 4 is also known as the chain rule: the derivative of the composite function is the derivative of the exterior function (here, f) multiplied by the derivative of the interior function (here, g). In the differentiation of f, the independent variable is not

x, but the argument of the function f, i.e., $g(x)$.

Example 6: Assuming that e^x is differentiable, show that e^{x^2+x-3} is differentiable:

We assign $f(y) = e^y$ and $g(x) = x^2+x-3$. g is a polynomial and as such differentiable. f is differentiable by hypothesis and therefore Theorem 4 establishes that e^{x^2+x-3} is differentiable, with the derivative given by:

$$(e^{x^2+x-3})' = e^{x^2+x-3}(2x+1).$$

Important: From this point on, we have all the tools to determine the derivatives of the fundamental functions and the functions obtained through composition and artithmetic operations of fundamental functions. In the Aula Virtual there is a list of the most relevant derivatives.

5.4 Fundamental theorems of differentiable functions

In this section, we consider a series of theorems that represent fundamental properties on differentiable functions. These results are very useful in the application of derivatives in the graphical discussion of the curves of functions. For that, we need to define concepts like extrema, critical points etc.

Definition 4 (local minimum and maximum):
A function f has a local (or relative) minimum in $x = c$ if there exists a $\delta > 0$ such that for all x that verify $|x - c| < \delta$ (with $x \in \mathrm{dom}(f)$), $f(x) \geq f(c)$ is fulfilled.

A function f has a local (or relative) maximum in $x = c$ if there exists a $\delta > 0$ such that for all x that verify $|x - c| < \delta$ (with $x \in \mathrm{dom}(f)$), $f(x) \leq f(c)$ is fulfilled.

Comments:
(1) Minima and maxima are *extrema*.

(2) It is important that *one* value of δ is sufficient, we do not require that the inequality for $f(x)$ is fulfilled *for all* δ. In practice, δ can be chosen as small as necessary.

(3) According to this definition, for functions which are constant in an interval I, *all* points $x \in I$ are at the same time local maxima and minima.

With that definition – that only uses the values of $f(x)$ in a neighborhood – to characterize a local extremum, we can formulate a fundamental theorem for local extrema.

Theorem 5 (theorem of local extrema):
If f is differentiable in $x = c$ and has a local extremum in $x = c$, then $f'(c) = 0$.

Comments:
(1) $f'(c) = 0$ implies that the tangent in $(c, f(c))$ is horizontal.

(2) This theorem does not hold if the function is only left or right differentiable in $x = c$, as the following example shows: Consider $f(x) = x^2$ and the closed interval $[1, 2]$. In that interval, $f(x)$ has local extrema in $x = 1$ (minimum) and $x = 2$ (maximum) according to Def. 4. But the function is only left differentiable in $x = 1$ and right differentiable in $x = 2$, so it is not (fully) differentiable in either point and we cannot apply the theorem. If we consider the same function in the open interval $(1, 2)$, then f is indeed (fully) differentiable for all $x \in (1, 2)$, but there are no local extrema in $(1, 2)$, so the theorem cannot be applied either. By considering only differentiable functions on open intervals, we avoid this kind of situations.

(3) The converse of the theorema does not hold, i.e., $f'(c) = 0$ does not imply that there is a local extremum in $x = c$. Example: The function is $f(x) = x^3$. We find $f'(x) = 3x^2$ and hence $f'(0) = 0$ (with a horizontal tangent), but in $x = 0$ there is no extremal value as the curve of f is monotonously increasing.

The next theorem establishes the conditions which assure the existence of a point c with $f'(c) = 0$:

Theorem 6 (Rolle's theorem):
Let f be a continuous function in an interval $[a, b]$, differentiable in (a, b) and that verifies $f(a) = f(b)$. Then, there exists at least one point $c \in (a, b)$ with $f'(c) = 0$.

Figure 2 illustrates Rolle's theorem.
The outline of the proof is as follows: As f is continuous in $[a, b]$, it attains its extrema in $[a, b]$, i.e., due to the boundedness property of functions there exist d, e in $[a, b]$ such that $f(e) \leq f(x) \leq f(d)$. If $f(e) = f(d)$, then f is a constant function in the interval and therefore verifies $f'(x) = 0$ for all values of the interval. If $f(e) \neq f(d)$, then at least one of e or d does not coincide with a or b (since $f(a) = f(b)$). Then, f has to have at least one extremum in $[a, b]$. As consequence of the theorem of local extrema, f' is zero in at least one point c of the interval, i.e., $f'(c) = 0$.

Comment:
The theorem establishes that there is a point c with $f'(c) = 0$, but is also assures the existence of a local extremum. Recall that according to Def. 4 all points of a constant function are minima and maxima at the same time. And for non-constant continuous functions there must be increasing and decreasing parts. Alternative formulations of Rolle's theorem would only ensure extrema for non-constant functions. In neither case there is a contradiction to what has been said in example (b) of comment (3) of Theorem 5 because that function does not fulfill $f(a) = f(b)$ (on any interval $[a, b]$).

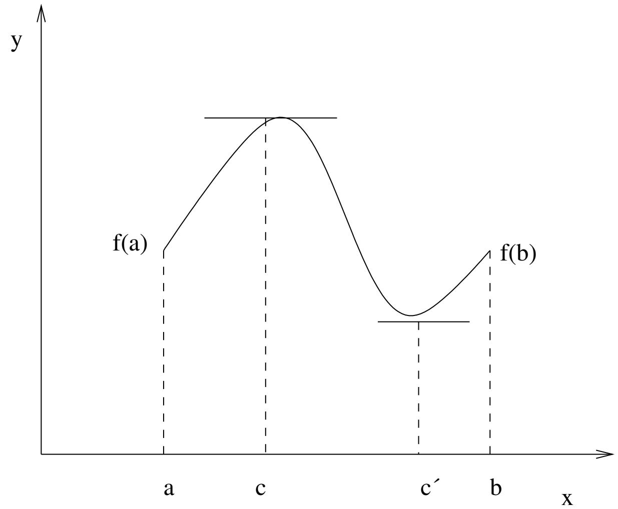

Figure 2: Illustration of Rolle's theorem.

Example 7:
The function $f(x) = x^3 - 12x$, with $x \in [0, 2\sqrt{3}]$, is a polynomial. As such, it is differentiable and continuous in $\mathbb{R}$, in particular in $[0, 2\sqrt{3}]$. It verifies $f(0) = f(2\sqrt{3}) = 0$ and hence all hypotheses of Rolle's theorem are met. Consequently, there exists (at least one) $c \in (0, 2\sqrt{3})$ for which $f'(c) = 0$. Actually, $f'(x) = 3x^2 - 12$, and hence $f'(c) = 0$ implies $c = \pm 2$. Only $c = 2$ lies within the interval under consideration and therefore represents the value verifying Rolle's theorem.

Theorem 7 (mean value theorem, MVT):
Let f be a continuous function in $[a, b]$ and differentiable in (a, b). Then, there exist at least one point $c \in (a, b)$ with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

For the graphical interpretation of the theorem, we go back to Figure 1. If we identify the point A with $(a, f(a))$ and P with $(b, f(b))$, then $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the secant passing through A and P. The MVT states that there is a value $c \in (a, b)$ such that in the point $(c, f(c))$ of the curve $f(x)$ the slope of the tangent, i.e., $f'(c)$, has the same value as $\dfrac{f(b) - f(a)}{b - a}$. In other words, there is at least a tangent to $f(x)$ parallel to the secant passing through $(a, f(a))$ and $(b, f(b))$. This is also illustrated in Figure 3.

Comment:
To justify the name of the theorem, we consider the following example. If the variable x represents time and $f(x)$ represents the position of a body in motion, $f(b) - f(a)$ represents the distance traveled in the interval of time $b - a$. Therefore, the quotient

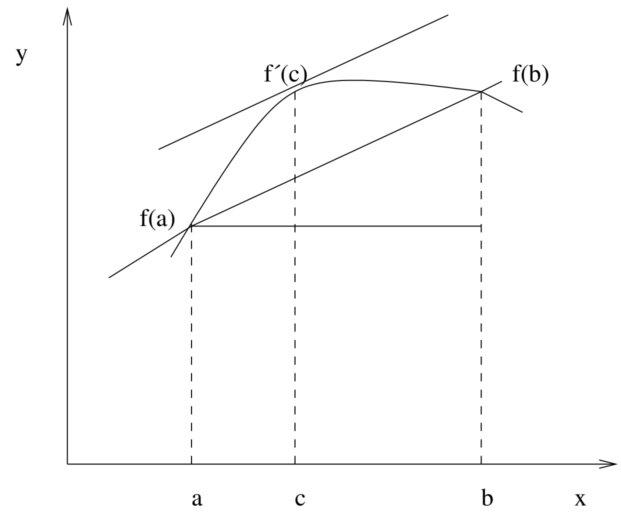

Figure 3: Illustration of the mean value theorem.

$\dfrac{f(b)-f(a)}{b-a}$ can be interpreted as the mean (average) velocity of the body in the considered time interval. The mean value establishes that there is at least one time moment of the interval when the instantaneous velocity of the body coincides with the mean velocity of the body in the whole interval. Note the similarity with the intermediate value theorem for continuous functions. This example also shows a physical interpretation of the derivative: the velocity is the derivative of the position with respect to time.

Outline of the proof:
Due to the similarity to Rolle's theorem in hypotheses and conclusion, we want to use it in the proof. But for doing so, we need a function, say g, that verifies $g(a) = g(b)$. Let us define $g(x)$ as

$$g(x) = f(x) - \gamma x, \quad \gamma \in \mathbb{R},$$

where $f(x)$ is the function of the statement of the MVT, i.e., it is differentiable in (a, b) and continuous in $[a, b]$. The term $-\gamma x$ represents a differentiable (and continuous) function in $\mathbb{R}$ and therefore $g(x)$ is also differentiable in (a, b) and continuous in $[a, b]$. To use Rolle's theorem, we need to fix γ such that $g(a) = g(b)$. From

$$\begin{aligned} g(a) &= f(a) - \gamma a, \\ g(b) &= f(b) - \gamma b, \end{aligned}$$

we get $f(a) - \gamma a = f(b) - \gamma b$ and therefore

$$\gamma = \frac{f(b) - f(a)}{b - a}.$$

Now, $g(x) = f(x) - \dfrac{f(b)-f(a)}{b-a} \cdot x$ verifies the hypothesis of Rolle's theorem and therefore there exists a $c \in (a, b)$ such that $g'(c) = 0$.

$$g'(c) \;=\; f'(c) - \frac{f(b) - f(a)}{b - a} = 0,$$

and, solving for $f'(c)$, we obtain the conclusion of the MVT:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 8: Show that $|\sin b - \sin a| \leq |b - a|$.

The function $f(x) = \sin x$ is differentiable and continuous in $\mathbb{R}$. Hence, it fulfills the hypotheses of the MVT in the interval $[a, b] \subset \mathbb{R}$ and there exists a point $c \in (a, b)$ such that

$$\frac{\sin b - \sin a}{b - a} = f'(c) = \cos c.$$

The cosine function is bounded:

$$\left|\frac{\sin b - \sin a}{b - a}\right| = |\cos c| \leq 1$$

and we conclude

$$|\sin b - \sin a| \leq |b - a|.$$

Theorem 8 (Cauchy's mean value theorem, CMVT):
Let f and g be continuous functions in the interval $[a, b]$ and differentiable in (a, b). Then, there exists a point $c \in (a, b)$ such that $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$.

Comments:
(1) This theorem can be seen as a generalization of the MVT for two functions. The main idea of the proof is to define a function $h(x) = g(x)(f(b)-f(a))-f(x)(g(b)-g(a))$ and subsequently use Rolle's theorem. Then, the justification is analogous to the MVT. It is left as an exercise.

(2) Let $g'(x) \neq 0$ for all $x \in (a, b)$. Then, CMVT can be formulated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)},$$

a useful version to show L'Hôpital's rule.

Theorem 9 (L'Hôpital's rule for $0/0$):
Let $x_0 \in (a, b)$ and f and g continuous functions in $[a, b]$ and differentiable in (a, b). If $f(x_0) = g(x_0) = 0$ and $\lim_{x \to x_0} \dfrac{f'(x)}{g'(x)} = L$, then $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = L$.

Outline of the proof:
The CMVT establishes the existence of a point c in an interval in which it verifies $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$ (if $g'(x) \neq 0$ for the points of the interval). L'Hôpital's rule considers the limit as $x \to x_0$, which is different from c. Indeed, we consider two intervals separately, $[x, x_0]$ (verifying for x: $a \leq x < x_0$) and $[x_0, x]$ (verifying for x: $x_0 < x \leq b$) and use the CMVT in both intervals.

Let us first consider the interval $[x_0, x]$ (verifying for x: $x_0 < x \leq b$). In this interval, f and g verify the hypotheses of the CMVT and hence there existes a point $c \in (x_0, x)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f(x)}{g(x)},$$

where we have used that $f(x_0) = g(x_0) = 0$. We have almost reached our conclusion as we already have established a relationship between f'/g' and f/g. To finalize the argument, we have to take the limit $x \to x_0+$. As c is between x and x_0, this implies that also $c \to x_0+$:

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = \lim_{c \to x_0+} \frac{f'(c)}{g'(c)} = \lim_{x \to x_0+} \frac{f'(x)}{g'(x)} = L$$

according to the hypothesis. For the interval $[x, x_0]$ the procedure is analogous: there exists a point $c \in (x, x_0)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f(x)}{g(x)}.$$

We take the limit $x \to x_0-$. As c is between x and x_0, this implies that also $c \to x_0-$:

$$\lim_{x \to x_0-} \frac{f(x)}{g(x)} = \lim_{c \to x_0-} \frac{f'(c)}{g'(c)} = \lim_{x \to x_0-} \frac{f'(x)}{g'(x)} = L,$$

with the same value L, i.e., the two lateral limits exist and coincide and as a consequence, the limit exists, with

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L.$$

Example 9: Calculate $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$ with L'Hôpital's rule.

Here, $f(x) = \sin x$ and $g(x) = x$ and the limit $x \to 0$ creates an indetermination $0/0$. f and g are differentiable in $\mathbb{R}$ and $f(0) = g(0) = 0$. Furthermore, $g'(x) = 1 \neq 0$ for all x. Consequently, the hypotheses of Theorem 9 are met and we calculate

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

Example 10: Calculate $\lim_{x \to 0} \dfrac{x - \sin x}{1 - e^x}$ with L'Hôpital's rule.

Here, $f(x) = x - \sin x$ y $g(x) = 1 - e^x$ and the limit $x \to 0$ creates an indetermination $0/0$.f and g are differentiable in $\mathbb{R}$ and $f(0) = g(0) = 0$. Furthermore, $g'(x) = -e^x \neq 0$ for all x. Consequently, the hypotheses of Theorem 9 are met and we calculate

$$\lim_{x \to 0} \frac{x - \sin x}{1 - e^x} = \lim_{x \to 0} \frac{1 - \cos x}{-e^x} = \frac{1 - \cos 0}{1} = \frac{0}{1} = 0.$$

Let us formulate L'Hôpital's rule for the general case, which not only includes indeterminations $0/0$ but also ∞/∞:

Theorem 10 (general L'Hôpital's rule):
Let $c \in (a, b)$ and f and g be differentiable functions in $(a, b) \setminus \{c\}$. If $g'(x) \neq 0$ for $c \in (a, b)$ and either $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty$, and $\lim_{x \to c} \dfrac{f'(x)}{g'(x)} = L$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = L$

Comment:
It is even possible to extend the rule to $x \to \pm\infty$ or if $L \to \pm\infty$.